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Time-Dependent Correlations for a One-Component Plasma in a Uniform Magnetic Field

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We derive various sum rules for the time-displaced structure function of a classical one-component plasma subjected to an external uniform magnetic field. When the plasma has some translational invariance (i.e., homogeneous or translation-invariant along the field), we find that there are long-wavelength oscillations with well-defined frequencies. The results are obtained from linear response and macroscopic electrodynamics, as well as from the microscopic equations of motion (BBGKY hierarchy). In the presence of the magnetic field, the time-displaced structure function has a polynomial decay at large distances, even in the homogeneous case. When the plasma has no translational invariance, examples show a more complicated temporal behaviour in the long-length-scale limit, involving a superposition of oscillations over a continuous range of frequencies.

KEY WORDS: Sum rules; time-dependent correlations; one-component plasma; magnetic field.

1. INTRODUCTION

The purpose of this paper is to generalize sum rules about the time-dependent correlations in a one-component plasma to the case when the plasma is submitted to an external uniform magnetic field. The presence of a magnetic field brings in new theoretically interesting features, and it should also be noted that experiments on one-component plasmas in magnetic fields are being performed.[1,2]

A one-component plasma is a system of identical particles of charge e and mass m in a fixed background of opposite charge. For simplicity, we

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use the framework of classical mechanics (generalization to quantum mechanics would be easy). The time-dependent charge correlation function (structure function) is defined as

$$S(q, t \mid q_1) = e^2[\langle N(q, t)\, N(q_1, 0)\rangle - \langle N(q, t)\rangle\langle N(q_1, 0)\rangle] \quad (1.1)$$

where $eN(q, t)$ is the microscopic charge density at point q and time t and $\langle\cdots\rangle$ is the equilibrium average at inverse temperature β.

In the case without a magnetic field, S was shown to obey a rather general sum rule involving a very simple oscillatory behavior[3,4]

$$\beta \int dq \int dq_1 \frac{1}{|q_1|} S(q, t \mid q_1) = \cos \bar{\omega} t \quad (1.2)$$

For a homogeneous plasma, $\bar{\omega}$ is the plasma frequency and (1.2) is equivalent to a well-known long-wavelength sum rule. It is remarkable that (1.2) also holds for a large class of inhomogeneous plasmas when the position-dependent background charge density $e\rho_b(q)$ has at infinity a limit $e\rho_\infty(\Omega)$, which may, however, depend upon the direction Ω in which q recedes to infinity; then $\bar{\omega}^2$ is the angular average of the square plasma frequency at infinity:

$$\bar{\omega}^2 = \frac{e^2}{m} \int d\Omega\, \rho_\infty(\Omega) \quad (1.3)$$

In the present paper, we wish to generalize (1.2) to the case when there is an external uniform magnetic field [of course, at $t=0$, (1.2) will be simply unchanged, since there are no static magnetic effects in classical statistical mechanics (Bohr–Van Leeuwen theorem)]. It will be found that a simple result, more or less similar to (1.2), is obtained, only for a more restricted class of cases, when the plasma is translationally invariant along the field (this includes the simplest case of homogeneous plasma). In other situations, it may still be possible to compute the left-hand side of (1.2), but the result is neither simple nor of a general form.

Even in the simplest case of a homogeneous plasma, it is already clear that the integral on the left-hand side of (1.2) requires some further specification in order to have a meaning. In the presence of a uniform magnetic field, $S(q, t \mid q_1)$, considered as a function of q, is a charge distribution which carries no net total charge (at $t=0$, this statement just expresses perfect screening, and it remains true at $t \neq 0$ by charge conservation); however, S is expected to carry an electrical quadrupole moment (the lowest multipole moment compatible with the symmetry of the system). Therefore,

$$\int dq_1 \frac{1}{|q_1|} S(q, t \mid q_1)$$

behaves like the potential created at the origin by a quadrupole moment centered at q, and this integral decays only as $|q|^{-3}$ for large q; the final integral upon q is not absolutely convergent. We perform this integral on some finite domain Λ, defining

$$I_\Lambda(t) = \int_\Lambda dq \int dq_1 \frac{1}{|q_1|} S(q, t \mid q_1) \tag{1.4}$$

and we shall show in the following that there are different ways of taking the limit $|\Lambda| \to \infty$, which give different values for $\lim I_\Lambda$.

As in Ref. 4, we shall first derive the sum rules by using a linear response argument (Section 2), taking for granted that macroscopic physics is valid on long length scales (the special thing about one-component plasmas is that long-wavelength charge oscillations are not damped). In Section 3, the problem is investigated from a microscopic point of view, based upon the BBGKY hierarchy and reasonable clustering assumptions.

In the homogeneous OCP (or when there is at least translational invariance along the magnetic field), the microscopic long-wavelength temporal behavior agrees exactly with that predicted by macroscopic electrodynamics. This is because the BBGKY hierarchy has an exact closure in this limit, which relies on sum rules valid for charged systems. We have, however, not been able to extend the theorem to more general inhomogeneous systems, due to the lack of information on the spatial decay of the correlations in complicated cases.

2. LINEAR RESPONSE APPROACH

2.1. Method

The approach used in this section relies upon the assumption that the linear response of the plasma to an external charge is correctly described for long length scales by a simple macroscopic dielectric tensor. The information about the response is converted into information about the correlations through the use of the fluctuation-dissipation theorem.[5]

When a system that was in equilibrium is subjected to a perturbation described by a Hamiltonian $A \cos \omega t$, where A is a dynamical variable, the change of the average value of a dynamical variable B is of the form $\mathrm{Re}[\chi_{BA}(\omega) \exp(-i\omega t)]$, to first order in A; this defines the response function $\chi_{BA}(\omega)$. On the other hand, in the unperturbed system in equilibrium, we can define a time-dependent correlation function

$$C_{BA}(t) = \langle B(t)\, A(0) \rangle - \langle B(t) \rangle \langle A(0) \rangle \tag{2.1}$$

and its Fourier transform

$$\tilde{C}_{BA}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt\, C_{BA}(t) \exp(i\omega t) \tag{2.2}$$

The fluctuation-dissipation theorem,[5] which relates χ_{BA} and $\tilde{C}_{BA}$, is, in its classical version,

$$\omega \tilde{C}_{BA}(\omega) = -(1/\pi\beta) \operatorname{Im} \chi_{BA}(\omega) \tag{2.3}$$

This simple form holds under the assumption that A and B have the same parity under time reversal and that χ_{BA} is unchanged by a reversal of the magnetic field.

Here, we choose the perturbation as caused by an external oscillating point charge of magnitude $\cos \omega t$ located at the origin, and the variable B as the charge density at point q; therefore

$$A = e \int dq_1 \frac{1}{|q_1|} N(q_1) \tag{2.4}$$

$$B = B(q) = eN(q) \tag{2.5}$$

The charge density induced at q, i.e., the change of the statistical average of $eN(q)$, is of the form $\operatorname{Re}[\rho(q, \omega) \exp(-i\omega t)]$ and the fluctuation-dissipation theorem (2.3) gives

$$\int dq_1 \frac{1}{|q_1|} S(q, t \mid q_1) = -\frac{1}{\pi\beta} \int_{-\infty}^{+\infty} \frac{d\omega}{\omega} \operatorname{Im} \rho(q, \omega) \exp(-i\omega t) \tag{2.6}$$

and by integration on a domain Λ,

$$\begin{aligned} I_\Lambda(t) &= \int_\Lambda dq \int dq_1 \frac{1}{|q_1|} S(q, t \mid q_1) \\ &= -\frac{1}{\pi\beta} \int_{-\infty}^{+\infty} \frac{d\omega}{\omega} \operatorname{Im} Q_\Lambda(\omega) \exp(-i\omega t) \end{aligned} \tag{2.7}$$

where

$$Q_\Lambda(\omega) = \int_\Lambda dq\, \rho(q, \omega) \tag{2.8}$$

represents the total charge induced in Λ. Therefore, $I_\Lambda(t)$ can be obtained from $Q_\Lambda(\omega)$, which will now be computed in a variety of cases.

2.2. Homogeneous Plasma

We consider a homogeneous plasma: the background charge density has the uniform value $-e\rho_b$. There is an external uniform magnetic field B. In the presence of an external charge density of the form $\mathrm{Re}[\rho_{\mathrm{ext}}(q)\exp(-i\omega t)]$, the linearized equations of motion on a macroscopic scale are

$$\nabla^2\phi = -4\pi(\rho + \rho_{\mathrm{ext}}) \tag{2.9a}$$

$$-i\omega j = -\frac{e^2}{m}\rho_b\nabla\phi + \frac{e}{m}j \wedge B \tag{2.9b}$$

$$i\omega\rho = \nabla\cdot j \tag{2.9c}$$

where ϕ, ρ, and j represent the induced electrical potential, charge density, and current density, respectively, in the usual complex representation (the actual electrical potential is $\mathrm{Re}[\phi\exp(-i\omega t)]$, etc.). Combining Eqs. (2.9a)–(2.9c) and using Cartesian coordinates $q=(x, y, z)$ with the z axis parallel to the magnetic field B, we obtain

$$\varepsilon_{\parallel}\frac{\partial^2\phi}{\partial z^2} + \varepsilon_{\perp}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) = -4\pi\rho_{\mathrm{ext}} \tag{2.10}$$

where

$$\varepsilon_{\parallel} = \frac{\omega^2 - \omega_p^2}{\omega^2}, \qquad \varepsilon_{\perp} = \frac{\omega^2 - \omega_p^2 - \omega_c^2}{\omega^2 - \omega_c^2} \tag{2.11}$$

are the well-known parallel and perpendicular dielectric functions[6]; ω_p and ω_c are the plasma and cyclotron frequencies defined by

$$\omega_p^2 = 4\pi(e^2/m)\rho_b, \qquad \omega_c = eB/m \tag{2.12}$$

The total charge $Q_\Lambda(\omega)$ induced in a *large* domain Λ by a point charge at the origin must be correctly given by taking $\rho_{\mathrm{ext}} = \delta(q)$, solving (2.10) for ϕ, obtaining ρ from (2.9a), and using it in (2.8). This is because Q_Λ is sensitive only to the long-wavelength space Fourier components of $\rho(q, \omega)$, and they are correctly given by the macroscopic approach, although the details of $\rho(q, \omega)$ are not. Actually, although this program can be carried on in position space, it is simpler to work with Fourier transforms. The result for

$$\tilde{\rho}(k, \omega) = \int dq\, e^{-ik\cdot q}\rho(q, \omega) \tag{2.13}$$

is valid in the small-k limit, and must be written as

$$\lim_{|k| \to 0} \tilde{\rho}(|k|, \theta, \omega) = \frac{\omega^2 - \omega_c^2}{\omega^2 - \omega_p^2 - \omega_c^2 + (\omega_p^2 \omega_c^2/\omega^2) \cos^2 \theta} - 1 \qquad (2.14)$$

where θ is the angle between the field B and the wave vector k. It should be noted that this limit depends upon the direction θ of k. As a consequence, as $|\Lambda| \to \infty$, the behavior of $Q_\Lambda(\omega)$ depends upon the way in which Λ becomes large.

Let us choose for Λ a cylinder of radius R and length $2L$, with its center at the origin and its axis u making an angle θ with B. Then, the "wide-cylinder" limit of Q_Λ is (2.14):

$$\lim_{L \to \infty} \lim_{R \to \infty} Q_\Lambda(\omega) = \lim_{|k| \to 0} \tilde{\rho}(|k|, \theta, \omega) \qquad (2.15)$$

since taking the first limit $R \to \infty$ amounts to taking k along u in (2.13). With ω understood as having an infinitesimal positive imaginary part which ensures that the perturbation is introduced adiabatically, after some algebra we find, from (2.7), (2.8), (2.14), and (2.15), the generalization of (1.2) to the present case:

$$\beta \lim_{L \to \infty} \lim_{R \to \infty} I_\Lambda(t) = \frac{1}{\omega_+^2 - \omega_-^2} \left[(\omega_+^2 - \omega_c^2) \cos \omega_+ t - (\omega_-^2 - \omega_c^2) \cos \omega_- t \right] \qquad (2.16)$$

where ω_+ and ω_- are the poles of $\tilde{\rho}$, i.e.,

$$\omega_\pm^2 = \tfrac{1}{2} \{ \omega_p^2 + \omega_c^2 \pm [(\omega_p^2 + \omega_c^2)^2 - 4\omega_p^2 \omega_c^2 \cos^2 \theta]^{1/2} \} \qquad (2.17)$$

In the special case $\theta = 0$, the right-hand side of (2.16) is simply $\cos \omega_p t$; in the special case $\theta = \pi/2$, it is

$$\frac{\omega_p^2}{\omega_p^2 + \omega_c^2} \cos[(\omega_p^2 + \omega_c^2)^{1/2} t] + \frac{\omega_c^2}{\omega_p^2 + \omega_c^2} \qquad (2.18)$$

At $t = 0$, the result is always 1, independent of B (no static magnetic effect). For $B = 0$, one recovers $\cos \omega_p t$, i.e., (1.2).

In the case where the cylinder axis is parallel to B, we can also compute the "long-cylinder" limit $\beta \lim_{R \to \infty} \lim_{L \to \infty} I_\Lambda(t)$; since this amounts to taking k normal to u, the result is (2.18), the same as for a "wide cylinder" with its axis normal to B.

In the present case of a homogeneous plasma, (2.16) can also be written in other forms, by taking advantage of the fact that $S(q, t \mid q_1)$ depends upon q and q_1 only through $q-q_1$. In terms of the Fourier-transformed structure function

$$\tilde{S}(k, t) = \int dq\, e^{-ik\cdot(q-q_1)} S(q, t \mid q_1) \tag{2.19}$$

(2.16) becomes

$$\tilde{S}(|k|, \theta, t) \underset{|k|\to 0}{\sim} \frac{k^2}{4\pi\beta} \frac{1}{\omega_+^2 - \omega_-^2} [(\omega_+^2 - \omega_c^2) \cos \omega_+ t - (\omega_-^2 - \omega_c^2) \cos \omega_- t] \tag{2.20}$$

The left-hand side of (2.16) can also be expressed as a second moment of S in position space:

$$\lim_{L\to\infty} \lim_{R\to\infty} I_A(t) = -2\pi \lim_{L\to\infty} \lim_{R\to\infty} \int_A dq\, q_u^2 S(q, t \mid 0) \tag{2.21}$$

where q_u is the component of q along the cylinder axis.

Equation (2.20) makes it apparent that the simple time dependence of the right-hand side of (2.16), involving only discrete frequencies ω_+ and ω_-, has been obtained because the limit of I_A has been taken in special ways that select one direction for k; only the two wave frequencies associated with that direction appear. Other ways of taking the limit will in general give whole frequency ranges. For instance, if we take for A a sphere of radius R centered at the origin, $\tilde{\rho}(k, \omega)$ first must be averaged on all the directions of k:

$$\lim_{R\to\infty} Q_A(\omega) = \lim_{|k|\to 0} \frac{1}{2} \int_{-1}^{1} d(\cos\theta)\, \tilde{\rho}(|k|, \theta, \omega) \tag{2.22}$$

From (2.7), (2.14), and (2.22), we now find

$$\beta \lim_{R\to\infty} I_A(t) = \int_0^{\inf(\omega_c, \omega_p)} d\omega\, g(\omega) \cos \omega t + \int_{\sup(\omega_c, \omega_p)}^{(\omega_p^2 + \omega_c^2)^{1/2}} d\omega\, g(\omega) \cos \omega t \tag{2.23}$$

where

$$g(\omega) = \frac{|\omega^2 - \omega_c^2|}{\omega_p \omega_c (\omega_p^2 + \omega_c^2 - \omega^2)^{1/2}} \tag{2.24}$$

The range of frequencies in (2.23) covers all the values given by (2.17) for any $\cos\theta$.

For large $|q|$, the asymptotic behavior of the induced charge density $\rho(q, \omega)$ is determined by the singularity (2.14) at $k=0$ (under the assumption that this singularity is the only one). Therefore, for obvious dimensional reasons, $\rho(q, \omega)$ decays like $|q|^{-3}$, and so does the left-hand side of (2.6), as foresen in the Introduction.

Similarly, the asymptotic behavior of $S(q, t \mid q_1)$ for large $|q|$ is determined by the singularity of $\tilde{S}(k, t)$ at $k=0$. Since, from (2.20), this singularity is of the form k^2 times a function of θ, $S(q, t \mid q_1)$ decays like $|q|^{-5}$. Let us remark that this algebraic decay would not be easily revealed by a small-t expansion of (2.20); in an expansion in powers of t^2, up to order t^6 the coefficients are regular functions of k (linear combinations of k^2 and $k^2 \cos^2 \theta = k_z^2$), and one has to go to order t^8 to see a singular term $k^2 \cos^4 \theta = k_z^4/k^2$ in the coefficient.

From (2.20), it is easy to obtain information about the charge–current correlation function, defined as

$$d(q, t \mid q_1) = e^2 \langle J(q, t)\, N(q_1, 0) \rangle \tag{2.25}$$

where eJ is the microscopic electrical curent density; the Fourier transform is

$$\tilde{d}(k, t) = \int dq \exp[-ik \cdot (q - q_1)]\; d(q, t \mid q_1) \tag{2.26}$$

For large length scales, $eJ(q, t)$ is related to $eN(q, t)$ by equations similar to (2.9) (with $\rho_{\text{ext}} = 0$), and the small-k behavior of $\tilde{d}(k, t)$ can be obtained from the small-k behavior of $\tilde{S}(k, t)$. In terms of the unit vector along the field $\hat{b} = B/|B|$, the result is

$$\begin{aligned} \tilde{d}(k, t) \underset{|k| \to 0}{\sim} \frac{-i\omega_p^2}{4\pi\beta} \frac{1}{\omega_+^2 - \omega_-^2} \bigg[& k\omega_+ \sin \omega_+ t \\ & - \hat{b}(\hat{b} \cdot k) \frac{\omega_c^2}{\omega_+} \sin \omega_+ t + (\hat{b} \wedge k)\omega_c \cos \omega_+ t - k\omega_- \sin \omega_- t \\ & + \hat{b}(\hat{b} \cdot k) \frac{\omega_c^2}{\omega_-} \sin \omega_- t - (\hat{b} \wedge k)\omega_c \cos \omega_- t \bigg] \end{aligned} \tag{2.27}$$

The singularity at $k=0$ is of the form k times a function of the polar angles of k, and therefore $d(q, t \mid q_1)$ decays like $|q|^{-4}$.

These algebraic decays of S and d are caused by the magnetic field. In the case $B=0$, the leading terms of $\tilde{S}$ and $\tilde{d}$ at $k=0$ are regular, respectively k^2 and k (of course, faster algebraic decays are not excluded by the present argument). On the contrary, if we consider the current–current correlation function, by similar methods we find for it an algebraic decay, like $|q|^{-3}$, already in the case $B=0$.

2.3. Inhomogeneous Plasma

We now allow the background charge density $-e\rho_b$ to be a function of position q (the magnetic field B, however, is again a uniform one). From the macroscopic equations (2.9) we now obtain, instead of (2.10),

$$\sum_{\alpha,\beta} \nabla_\alpha[\varepsilon_{\alpha\beta}\nabla_\beta\phi] = -4\pi\rho_{\text{ext}} \tag{2.28}$$

where α, $\beta = x$, y, z label the coordinate axes; z is along the field. The non-vanishing elements of the dielectric tensor[6] $\varepsilon_{\alpha\beta}$ are

$$\begin{aligned} \varepsilon_{zz} &= \varepsilon_{\|} = \frac{\omega^2 - \omega_p^2}{\omega^2} \\ \varepsilon_{xx} &= \varepsilon_{yy} = \varepsilon_{\perp} = \frac{\omega^2 - \omega_p^2 - \omega_c^2}{\omega^2 - \omega_c^2} \\ \varepsilon_{xy} &= -\varepsilon_{yx} = \frac{-i\omega_p^2\omega_c}{\omega(\omega^2 - \omega_c^2)} \end{aligned} \tag{2.29}$$

These elements are position-dependent through ω_p^2, which is defined by (2.12): their expressions (2.29) are valid in those regions of space where the variation of $\rho_b(q)$ is slow enough. The external charge density is chosen as a point source:

$$\rho_{\text{ext}} = \delta(q)$$

We were able to extract a sum rule from (2.28) only in special cases.

2.3.1. A Special Class of Inhomogeneous Plasma. In the presence of a magnetic field, we obtain a simple generalization of (1.2) for a restricted class of systems. Using cylindrical coordinates $q = (r, \varphi, z)$ with the z axis parallel to the uniform magnetic field, we choose the background charge density as a function $-e\rho_b(r, \varphi)$, independent of z, and we assume that ρ_b has, at infinity, a limit, which may, however, depend upon the direction in which r recedes to infinity: $\lim_{r\to\infty} \rho_b(r, \varphi) = \rho_\infty(\varphi)$ exists, for almost every φ.

The invariance along z suggests we introduce the Fourier transforms, with respect to z, of the potential ϕ and the induced charge density ρ:

$$\tilde{\phi}(r, \varphi, k, \omega) = \int dz\, e^{-ikz}\phi(r, \varphi, z, \omega) \tag{2.30}$$

$$\tilde{\rho}(r, \varphi, k, \omega) = \int dz\, e^{-ikz}\rho(r, \varphi, z, \omega) \tag{2.31}$$

Actually, we only need to consider the value $k=0$, at which (2.28) becomes

$$\sum_{\alpha,\beta=x,y} \nabla_\alpha[\varepsilon_{\alpha\beta}\nabla_\beta\tilde{\phi}(k=0)] = -4\pi\,\delta(x)\,\delta(y) \tag{2.32}$$

This is the equation for the potential created by a uniformly charged wire (the z axis), and $\tilde{\rho}(k=0)$ is the corresponding induced charge density. Alternatively, (2.32) defines a problem in two-dimensional electrostatics. The total induced charge in a large disk of radius R,

$$Q(\omega) = \int_0^R dr\, r \int_0^{2\pi} d\varphi\, \tilde{\rho}(r, \varphi, k=0, \omega) \tag{2.23}$$

can be obtained by adapting an argument of Ref. 4, as follows. In the asymptotic region (r large), where $\varepsilon_\perp(r, \varphi)$ and $\varepsilon_{xy}(r, \varphi)$ can be replaced by their asymptotic forms $\varepsilon_{\perp\infty}(\varphi)$ and $\varepsilon_{xy\infty}(\varphi)$, (2.32) becomes

$$\varepsilon_{\perp\infty}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\tilde{\phi}}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\varphi}\left(\varepsilon_{\perp\infty}\frac{\partial\tilde{\phi}}{\partial\varphi}\right) - \frac{1}{r}\left(\frac{d}{d\varphi}\varepsilon_{xy\infty}\right)\frac{\partial\tilde{\phi}}{\partial r} = 0 \tag{2.34}$$

with a solution of the form

$$\tilde{\phi} = -A[\ln r + f(\varphi)] \tag{2.35}$$

where $f(\varphi)$ is a function (periodic in φ with a period 2π) determined by (2.34). Through the circle of radius R, the radial component of the electrical displacement

$$-\varepsilon_{\perp\infty}(\partial\tilde{\phi}/\partial r) - \varepsilon_{xy\infty}(1/r)(\partial\tilde{\phi}/\partial\varphi)$$

has a flux

$$A\int_0^{2\pi} d\varphi\, \varepsilon_{\perp\infty}(\varphi) = 4\pi \tag{2.36}$$

while the radial component of the electrical field $-(\partial\tilde{\phi}|\partial r)$ has a flux

$$2\pi A = 4\pi[1 + Q(\omega)] \tag{2.37}$$

Since taking $k=0$ in (2.31) amounts to integrating $\rho(r, \varphi, z, \omega)$ with respect to z, $Q(\omega)$ is the charge induced in a cylinder Λ of length $2L$, radius R, and axis z in the "long-cylinder" limit. From (2.36) and (2.37), we find

$$\lim_{R\to\infty}\lim_{L\to\infty} Q_\Lambda(\omega) = 1/\bar{\varepsilon} - 1 \tag{2.38}$$

where $\bar{\varepsilon}$ is defined as an average at infinity of $\varepsilon_\perp$:

$$\bar{\varepsilon} = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, \varepsilon_{\perp\infty}(\varphi) = \frac{\omega^2 - \bar{\omega}_p^2 - \omega_c^2}{\omega^2 - \omega_c^2} \tag{2.39}$$

and

$$\bar{\omega}_p^2 = \frac{2e^2}{m} \int_0^{2\pi} d\varphi \, \rho_\infty(\varphi) \tag{2.40}$$

We can now use (2.38) in (2.7). It must be noted, however, that the general form (2.3) of the fluctuation-dissipation theorem allows us to add to $\tilde{C}_{BA}(\omega)$ a term proportional to $\delta(\omega)$, i.e., to add a constant to $\tilde{C}_{BA}(t)$. This constant is determined by the requirement that $C_{BA}(t=0)$ has the right value. Here, this procedure gives

$$\beta \lim_{R\to\infty} \lim_{L\to\infty} I_A(t) = \frac{\bar{\omega}_p^2}{\bar{\omega}_p^2 + \omega_c^2} \cos[(\bar{\omega}_p^2 + \omega_c^2)^{1/2} t] + \frac{\omega_c^2}{\bar{\omega}_p^2 + \omega_c^2} \tag{2.41}$$

This is the generalization of the sum rule (1.2) to the present class of systems.

Special cases of (2.41) include a semi-infinite plasma bounded by a plane wall $x=0$ [$\rho_b(q)=0$ if $x<0$, $\rho_b(q)=\rho_b$ if $x>0$], with the magnetic field parallel to the wall; then $\bar{\omega}_p = \omega_p/\sqrt{2}$, the surface plasma frequency. Another special case is a two-density plasma [$\rho_b(q)=\rho_-$ if $x<0$ and $\rho_b(q)=\rho_+$ if $x>0$), again with the magnetic field parallel to the plane $x=0$; then $\bar{\omega}_p = [(2\pi e^2/m)(\rho_+ + \rho_-)]^{1/2}$.

2.3.2. More Complicated Cases. An example: Solving (2.28) in less special geometries will in general result in a sum rule involving whole frequency ranges, as in (2.23) and (2.24). We shall only consider one example: a semi-infinite plasma bounded by a plane wall $z=0$ [$\rho_b(q)=0$ if $z<0$ and $\rho_b(q)=\rho_b$ if $z>0$], with the magnetic field along the z axis, i.e., normal to the wall.

With $\rho_{\text{ext}} = \delta(q)$, (2.28) becomes, in the present case,

$$\frac{\partial}{\partial z}\left(\varepsilon_{\|} \frac{\partial \phi}{\partial z}\right) + \varepsilon_{\perp}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) = -4\pi\, \delta(x)\, \delta(y)\, \delta(z) \tag{2.42}$$

$\varepsilon_{\|}$ and $\varepsilon_{\perp}$ are given by (2.29) for $z>0$, and are equal to 1 for $z<0$. Now, the invariance along the xy plane suggests we introduce the Fourier transforms with respect to $r=(x, y)$

$$\tilde{\phi}(k, z, \omega) = \int dr \, e^{-ik\cdot r} \phi(x, y, z, \omega) \tag{2.43}$$

$$\tilde{\rho}(k, z, \omega) = \int dr \, e^{-ik\cdot r} \rho(x, y, z, \omega) \tag{2.44}$$

Equation (2.42) becomes

$$\frac{\partial}{\partial z}\left(\varepsilon_{\|}\frac{\partial\tilde{\phi}}{\partial z}\right)-\varepsilon_{\perp}k^2\tilde{\phi}=-4\pi\delta(z) \tag{2.45}$$

It is straightforward to solve (2.45) in each region $z>0$ and $z<0$, with the conditions that $\tilde{\phi}$ is continuous and $\varepsilon_{\|}(\partial\tilde{\phi}/\partial z)$ has a jump -4π at $z=0$. The result is

$$\begin{aligned}\tilde{\phi}&=\frac{4\pi}{[1+(\varepsilon_{\|}\varepsilon_{\perp})^{1/2}]\,|k|}\exp\left[-\left(\frac{\varepsilon_{\perp}}{\varepsilon_{\|}}\right)^{1/2}|k|\,z\right], \qquad z>0\\ \tilde{\phi}&=\frac{4\pi}{[1+(\varepsilon_{\|}\varepsilon_{\perp})^{1/2}]\,|k|}\exp(|k|\,z), \qquad z<0\end{aligned} \tag{2.46}$$

From Gauss's theorem, we then obtain the total charge $Q(\omega)$ induced between the planes $z=\pm L$:

$$\begin{aligned}1+Q(\omega)&=\frac{1}{4\pi}\left[\left(\frac{\partial\tilde{\phi}}{\partial z}\right)_{z=L,k=0}-\left(\frac{\partial\tilde{\phi}}{\partial z}\right)_{z=-L,k=0}\right]\\ &=\frac{1+(\varepsilon_{\perp}/\varepsilon_{\|})^{1/2}}{1+(\varepsilon_{\perp}\varepsilon_{\|})^{1/2}}\end{aligned} \tag{2.47}$$

$Q(\omega)$ is the charge induced in a cylinder A of length $2L$, radius R, and axis z in the "wide-cylinder" limit, and can also be written as $\lim_{L\to\infty}\lim_{R\to\infty}Q_A(\omega)$.

In (2.46) and (2.47), the square roots $(\varepsilon_{\perp}\varepsilon_{\|})^{1/2}$ and $(\varepsilon_{\perp}/\varepsilon_{\|})^{1/2}$ must be defined as the positive square roots when $\varepsilon_{\|}$ and $\varepsilon_{\perp}$ are both positive, for instance, when $\omega^2>\omega_p^2+\omega_c^2$. For other values of ω, the square roots are defined by analytic continuation in the ω complex plane, with ω understood as having an infinitesimal positive imaginary part. In this way, we obtain Im $Q(\omega)$, and, from (2.7), a sum rule for the correlation function.

In the case $\omega_c<\omega_p$ the result is

$$\begin{aligned}\beta\lim_{L\to\infty}\lim_{R\to\infty}I_A(t)&=\cos\left[\left(\frac{\omega_p^2+\omega_c^2}{2}\right)^{1/2}t\right]+\int_0^{\omega_c}d\omega\,h(\omega)\cos\omega t\\ &\quad+\int_{\omega_p}^{(\omega_p^2+\omega_c^2)^{1/2}}d\omega\,h(\omega)\cos\omega t\end{aligned} \tag{2.48}$$

where

$$h(\omega)=\frac{2}{\pi}\left[\frac{(\omega_p^2+\omega_c^2-\omega^2)(\omega^2-\omega_c^2)}{\omega^2-\omega_p^2}\right]^{1/2}\frac{1}{\omega_p^2+\omega_c^2-2\omega^2} \tag{2.49}$$

The range of frequencies in (2.48) covers all the bulk wave frequencies given by (2.17) for any $\cos\theta$, plus the isolated frequency $[(\omega_p^2+\omega_c^2)/2]^{1/2}$, which corresponds to a surface wave. Similar results are obtained in the case $\omega_p<\omega_c$, with, however, no isolated frequency contribution. The "long-cylinder" limit $\beta \lim_{R\to\infty} \lim_{L\to\infty} I_A(t)$ can also be computed.

2.4. Dipole Sum Rules

For a semi-infinite plasma or a two-density plasma there are other sum rules, involving the dipole moment of the pair correlation function. These dipole sum rules can be obtained by considering the linear response to charged plates (rather than charged points). The derivation and results are simple generalizations of what was done in Ref. 4.

2.4.1. Semi-Infinite Plasma. We want to consider a semi-infinite plasma bounded by a plane wall $z=0$ $[\rho_b(q)=0$ if $z<0$ and $\rho_b(q)=\rho_b$ if $z>0]$ in a uniform magnetic field of arbitrary direction; we call θ the angle between the normal to the wall (the z axis) and the magnetic field. In a first step, we assume the plasma confined between walls at $z=0$ and $z=L$; the limit $L\to\infty$ will be taken afterward. The perturbation is caused by charging the walls at $z=0$ and $z=L$ with oscillating surface charge densities $\pm\cos\omega t$. Let the induced surface charge density along the wall $x=0$ be $\sigma(\omega)$. The electrical field is along the z axis, with a value $4\pi(1+\sigma)$ determined by the total surface charge density, while the z component of the electrical displacement is 4π, determined by the external surface charge density. The dielectric tensor relates the electrical displacement to the electrical field:

$$4\pi=(\varepsilon_{||}\cos^2\theta+\varepsilon_{\perp}\sin^2\theta)\,4\pi(1+\sigma) \tag{2.50}$$

Equation (2.50) determines $\sigma(\omega)$, which, from the microscopic point of view, is

$$\sigma(\omega)=\int_0^{L/2} dz\,\rho(z,\omega) \tag{2.51}$$

The Hamiltonian of the perturbation is $A\cos\omega t$, with

$$A=-4\pi e\int_{0<z_1<L} dq_1\, z_1 N(q_1) \tag{2.52}$$

Therefore, from the fluctuation-dissipation theorem (2.3), we find, after having taken the limit $L\to\infty$,

$$-4\pi\int_0^{\infty} dz\int dq_1\, z_1 S(q,t\,|\,q_1)=-\frac{1}{\pi\beta}\int_{-\infty}^{+\infty}\frac{d\omega}{\omega}\operatorname{Im}\sigma(\omega)\exp(-i\omega t) \tag{2.53}$$

The function $S(q, t \mid q_1)$ should have a slow decay (as $|q_1|^{-3}$) parallel to the wall, but as $z_1 \to \infty$, it is expected to decay as the bulk function (as $|q_1|^{-5}$); therefore the q_1 integral in (2.53) is convergent. Finally, from (2.50) and (2.29), we obtain the dipole sum rule:

$$-4\pi\beta \int_0^\infty dz \int dq_1\, z_1 S(q, t \mid q_1)$$

$$= \frac{1}{\omega_+^2 - \omega_-^2} \left[(\omega_+^2 - \omega_c^2) \cos \omega_+ t - (\omega_-^2 - \omega_c^2) \cos \omega_- t\right] \tag{2.54}$$

where ω_+ and ω_- are given by (2.17).

2.4.2. Two-Density Plasma. A very similar sum rule holds for a two-density plasma $[\rho_b(q) = \rho_-$ if $z < 0$ and $\rho_b(q) = \rho_+$ if $z > 0]$. The argument is the same as above, except for the fact that the induced charge density $\rho(z, \omega)$ now extends on both sides of the plane $z = 0$. Thus, we obtain a sum rule of the same form as (2.54), with $\int_0^\infty dz \cdots$ replaced by $\int_{-\infty}^\infty dz \cdots$; the integral on z_1 still is on positive z_1 only, and the frequencies $\omega_\pm$ still are those pertaining to the region $z > 0$, i.e., computed with $\rho_b = \rho_+$.

3. MICROSCOPIC THEORY

3.1. General Setting

We reinvestigate the derivation of sum rules from a microscopic viewpoint. The setting is the same as in Section 3 of Ref. 4, with the appropriate modifications to include the magnetic field. We assume that the time-dependent correlation functions of an OCP with background density $\rho_b(q)$ obey the BBGKY equation

$$\begin{aligned}\frac{\partial}{\partial t} \rho(q, v, t \mid U) = & -v \cdot \nabla_q \rho(q, v, t \mid U) \\ & - \frac{e}{m} [v \wedge B + E(q)]\, \nabla_v \rho(q, v, t \mid U) \\ & - \frac{e^2}{m} \int dq'\, F(q - q')\, \nabla_v [\rho(q, v; q', t \mid U) \\ & - \rho(q')\, \rho(q, v, t \mid U)]\end{aligned} \tag{3.1}$$

The correlation functions $\rho(q, v; q', v'; \ldots \mid U)$ between a set of particles with positions and velocities q, v; q', v';... at time t and another set of particles $U = (q_1, v_1; q_2, v_2; ...; q_k, v_k)$ at time $t = 0$ are defined by

$$\rho(q, v; q', v'; ..., t \mid U) = \langle [N(q, v, t)\, N(q', v', t) ...]_{nc} \times [N(q_1, v_1, 0) \cdots N(q_k, v_k, 0)]_{nc} \rangle \tag{3.2}$$

where $N(q, v, t)$ is the microscopic number density at point q, velocity v, and time t. The notation $[\cdots]_{nc}$ means that the contribution of coincident particles is not included, and $\langle \cdots \rangle$ is the thermal average. When the set U is empty, the correlations reduce to their equilibrium value, which are the static configurational correlations $\rho(q, q', ...)$ multiplied by the Maxwellian distribution of velocities. In (3.1) and in the sequel, the suppression of a velocity argument means that it has been integrated out. In particular, the structure function (1.1) is

$$S(q, t \mid q_1) = e^2(\rho(q, t \mid q_1) - \rho(q)\, \rho(q_1)] \tag{3.3}$$

In (3.1), $F(q) = -\nabla_q(1/|q|)$ is the Coulomb force and

$$E(q) = e \int dq'\, F(q - q')[\rho(q') - \rho_b(q')] \tag{3.4}$$

is the electric field due to the total charge density. When the particles are constrained to move in a restricted domain D bounded by hard walls, the configurational integrals are restricted to D and Eq. (3.1) is supplemented by the condition of elastic collisions at the walls.

We now proceed as in Section 3 of Ref. 4. We introduce the truncated correlations

$$\rho_T(q, v, t \mid U) = \rho(q, v, t \mid U) - \rho(q, v)\, \rho(U) \tag{3.5}$$

$$\rho_T(q, v; q', v', t \mid U) = \rho(q, v; q', v', t \mid U) - \rho(q, v)\, \rho(q', v', t \mid U) - \rho(q', v')\, \rho(q, v, t \mid U) - \rho(q, v; q', v')\, \rho(U) + 2\rho(q, v)\, \rho(q', v')\, \rho(U) \tag{3.6}$$

and assume that these functions have some decay properties in velocity and configuration space:

$$|\rho_T(q, v, t \mid U) \leqslant M/|v|^\eta, \qquad \eta > 5 \tag{3.7}$$

for fixed q, t, and U, and

$$|\rho_T(q, v, t \mid U)| \leqslant M/|q|^3 \tag{3.8}$$

for fixed v, t, and U.

The condition (3.7) ensures a sufficiently fast decay in velocity space at time t (at time $t=0$, the decay is Gaussian), whereas condition (3.8) is compatible with the findings of Section 2.2, where the current–current correlations behave as $|q|^{-3}$, $|q| \to \infty$. Other clustering assumptions will be introduced at the appropriate places.

After some algebra the BBGKY equation (3.1) yields for the truncated functions

$$\begin{aligned}\frac{\partial}{\partial t} \rho_T(q, v, t \mid U) = & -v \cdot \nabla_q \rho_T(q, v, t \mid U) \\ & -\frac{e}{m} [v \wedge B + E(q)] \cdot \nabla_v \rho_T(q, v, t \mid U) \\ & -\frac{e^2}{m} [\nabla_v \rho(q, v)] \cdot \int dq' \, F(q-q') \, \rho_T(q', t \mid U) \\ & -\frac{e^2}{m} \int dq' \, F(q-q') \cdot \nabla_v \rho_T(q, v; q', t \mid U) \end{aligned} \tag{3.9}$$

When we multiply (3.9) by a velocity-independent function and integrate on q and v, we get, after an integration by parts,

$$\frac{\partial}{\partial t} \int dq \, f(q) \, \rho_T(q, t \mid U) = \int dq \, [\nabla_q f(q)] \cdot \int dv \, v \rho_T(q, v, t \mid U) \tag{3.10}$$

Equations (3.9) and (3.10) have to be supplemented with initial conditions determined from the statics. At $t=0$

$$\begin{aligned} e\rho_T(q, t=0 \mid U) = e[\rho(q, q_1, ..., q_k) + \sum_{i=1}^{k} \delta(q-q_i) \, \rho(q_1, ..., q_k) \\ - \rho(q) \, \rho(q_1, ..., q_k)] \prod_{i=1}^{k} \left(\frac{\beta}{2\pi m}\right)^{3/2} \exp\left(-\beta \frac{1}{2} v_i^2\right) \end{aligned} \tag{3.11}$$

is the static excess charge density at q when particles at $U = (q_1, v_1; ...; q_k, v_k)$ are fixed. This excess charge satisfies the charge and dipole sum rules of Ref. 7:

$$e \int dq \, \rho_T(q, t=0 \mid U) = 0 \tag{3.12}$$

$$e \int dq \, q\rho_T(q, t=0 \mid U) = 0 \tag{3.13}$$

3.2. Homogeneous Plasma

In the homogeneous plasma, the local neutrality imposes $\rho(q)=\rho_b$ and $E(q)=0$. Because of translation invariance, it is convenient to introduce the Fourier transforms of the correlations. We consider specifically the correlations of the charge and of the current with the phase space densities of a general set of particles U,

$$e\int dq\, e^{-ik\cdot q}\rho_T(q, t\mid U)\equiv \tilde{C}(k, t\mid U) \tag{3.14}$$

$$e\int dq\, e^{-ik\cdot q}\int dv\, v\rho_T(q, v, t\mid U)\equiv \tilde{d}(k, t\mid U) \tag{3.15}$$

Notice that when $U=(q_1=0, v_1)$ reduces to a single particle and the integral on v_1 is performed, the quantities

$$e\int dv_1\, \tilde{C}(k, t\mid 0, v_1)=\tilde{S}(k, t) \tag{3.16}$$

$$e\int dv_1\, \tilde{d}(k, t\mid 0, v_1)=\tilde{d}(k, t) \tag{3.17}$$

are the Fourier transforms of the charge–charge correlation (2.19) and charge–current correlation (2.26). By homogeneity in space and time, one has

$$\rho_T(q, v, t\mid 0, v_1)=\rho(-q, v_1, -t\mid 0, v) \tag{3.18}$$

and this implies also with the definitions (3.14) and (3.15) that

$$\int dv_1\, v_1\tilde{C}(k, t\mid 0, v_1)=\tilde{d}(-k, -t) \tag{3.19}$$

One finds from (3.9) and (3.10) that the correlations (3.14) and (3.15) obey the equations

$$\frac{\partial}{\partial t}\tilde{d}(k, t\mid U)=\frac{e}{m}\tilde{d}(k, t\mid U)\wedge B-i\omega_p^2\frac{k}{|k|^2}\tilde{C}(k, t\mid U) \tag{3.20a}$$

$$+ie\int dv\,(k\cdot v)\, v\tilde{\rho}_T(k, v, t\mid U) \tag{3.20b}$$

$$+\frac{e^3}{m}\int dq\int dq'\, e^{-ik\cdot q}F(q-q')\,\rho_T(q, q', t\mid U) \tag{3.20c}$$

$$\frac{\partial}{\partial t}\tilde{C}(k, t\mid U)=-ik\cdot\tilde{d}(k, t\mid U) \tag{3.21}$$

If the terms (3.20b) and (3.20c) are neglected, Eqs. (3.20a) and (3.21) are analogous to the macroscopic equations (2.9) with ρ_{ext} set equal to zero, and $\tilde{C}(k, t \mid U)$ and $\tilde{d}(k, t \mid U)$ corresponding to the charge and current densities. Therefore, (3.20a) and (3.21) describe free macroscopic charge and current oscillations in the plasma. The terms (3.20b) and (3.20c) (a "kinetic energy" term and a "collision" term) incorporate the effects of the microscopic correlations. We show that these terms do not contribute to the long-wavelength limit of charge–charge and charge–current correlations. As a result, we will find that the behaviors (2.20) and (2.27) are exact in the microscopic theory.

3.2.1. Dynamics of the Correlations in the Long-Wavelength Limit. The small-k behavior of Eqs. (3.20) and (3.21) is investigated under the assumption that

$$\tilde{d}(k, t \mid U) = d_0(\hat{k}, t \mid U) + o(1), \qquad \hat{k} = k/|k| \tag{3.22}$$

i.e., $\tilde{d}(k, t \mid U)$ has a limit when $k \to 0$ in a fixed direction $\hat{k}$. Since $\tilde{d}(k, t \mid U)$ is the Fourier transform of the current correlated with the phase space densities of a general set of particles U, the assumption (3.22) is compatible with the spatial decay discussed in Section 2.2: for a general U, $\rho_T(q, v, t \mid U)$ should behave as a current–current corelation with a $|q|^{-3}$ decay. We deduce immediately from Eq. (3.21) that

$$\frac{\partial}{\partial t} \tilde{C}(k = 0, t \mid U) = \frac{\partial}{\partial t} \left[e \int dq\, \rho_T(q, t \mid U) \right] = 0 \tag{3.23}$$

With the initial condition (3.12) (the static charge sum rule), this implies that

$$\tilde{C}(k = 0, t \mid U) = e \int dq\, \rho(q, t \mid U) = 0 \tag{3.24}$$

for all t and U. We therefore conclude that the charge sum rule remains true for all times in the presence of the magnetic field.

We now write the small-$|k|$ expansion of $\tilde{C}(k, t \mid U)$ as

$$\tilde{C}(k, t \mid U) = C_1(\hat{k}, t \mid U)\,|k| + o(|k|) \tag{3.25}$$

and suppose, moreover,

$$\int dq \int dq' \,|\rho_T(q, q', t \mid U)| < \infty \tag{3.26}$$

The assumptions (3.25) and (3.26) are again compatible with the findings of Section 2.2: for a general U, $\rho(q, t \mid U)$ and $\rho(q, q', t \mid U)$ behave as charge–current correlations with a $|q|^{-4}$ decay.

Letting now $k \to 0$ in (3.20) and (3.21), we find

$$\frac{\partial}{\partial t} d_0(\hat{k}, t \mid U) = \frac{e}{m} d_0(\hat{k}, t \mid U) \wedge B - i\omega_p^2 \hat{k} C_1(\hat{k}, t \mid U) \tag{3.27}$$

$$\frac{\partial}{\partial t} C_1(\hat{k}, t \mid U) = -i\hat{k} \cdot d_0(\hat{k}, t, U) \tag{3.28}$$

Under the above assumptions, the terms (3.20b) and (3.20c) are $o(1)$ [the integrand of (3.20c) are antisymmetric in q, q' at $|k| = 0$].

Equations (3.27) and (3.28) form a closed set. One obtains by successive iterations

$$\frac{\partial^2}{\partial t^2} C_1(\hat{k}, t \mid U) = -i\omega_c \hat{k} \cdot [d_0(\hat{k}, t \mid U) \wedge \hat{b}] - \omega_p^2 C_1(\hat{k}, t \mid U) \tag{3.29}$$

$$\frac{\partial^3}{\partial t^3} C_1(\hat{k}, t \mid U) = i\omega_c^2 (\hat{k} - \cos\theta \hat{b}) \cdot d_0(\hat{k}, t \mid U) = \omega_p^2 \frac{\partial}{\partial t} C_1(\hat{k}, t \mid U) \tag{3.30}$$

with $\hat{b} = B/|B|$ and $\cos\theta = \hat{k} \cdot \hat{b}$, and finally

$$\left[\frac{\partial^4}{\partial t^4} + (\omega_p^2 + \omega_c^2) \frac{\partial^2}{\partial t^2} + \omega_c^2 \omega_p^2 \cos^2\theta\right] C_1(\hat{k}, t \mid U) = 0 \tag{3.31}$$

It is easily checked that the characteristic frequencies of Eq. (3.31) are precisely given by (2.17). Hence, $C_1(\hat{k}, t \mid U)$ [as well as $d_0(\hat{k}, t \mid U)$] is of the form

$$C_1(\hat{k}, t \mid U) = A \cos \omega_+ t + B \sin \omega_+ t + C \cos \omega_- t + D \sin \omega_- t \tag{3.32}$$

The coefficients have to be determined by the initial conditions, i.e., evaluating the equilibrium value $C_1(\hat{k}, t = 0 \mid U)$ as well as (3.28)–(3.30) at $t = 0$. Since the static truncated correlations have a fast decay, $C_1(\hat{k}, t = 0 \mid U)$ is given with (3.25) and (3.14) by the expression

$$C_1(\hat{k}, t = 0 \mid U) = -ie \int dq\, q \rho_T(q, t = 0 \mid U) = 0 \tag{3.33}$$

This vanishes because of the static dipole sum rule (3.13). The zero-time derivatives can be computed from

$$\begin{aligned} d_0(\hat{k}, t = 0 \mid U) &= e \int dq \int dv\, v \rho_T(q, v, t = 0 \mid U) \\ &= eV\rho(U) \end{aligned} \tag{3.34}$$

with $V=\sum_{i=1}^{k} v_i$. The final result is

$$C_1(\hat{k}, t \mid U) = e\rho(U) \frac{i}{\omega_+^2 - \omega_-^2} \left[\omega_c V \cdot (\hat{b} \wedge k)(\cos \omega_+ t - \cos \omega_- t) - (\hat{k} \cdot \hat{b})\hat{b} \cdot V\omega_c^2 \left(\frac{\sin \omega_- t}{\omega_-} - \frac{\sin \omega_+ t}{\omega_+} \right) + \hat{k} \cdot V(\omega_- \sin \omega_- t - \omega_+ \sin \omega_+ t) \right] \tag{3.35}$$

This is the exact small-k behavior of the $(n+1)$-point function (3.14). Two special cases are of interest. If one integrates (3.35) on $v_1, \ldots, v_k$, one gets [since $\rho(U)$ is Maxwellian]

$$C_1(\hat{k}, t \mid q_1, \ldots, q_k) = 0 \tag{3.36}$$

for all $t, q_1, \ldots, q_k$. When $\rho_T(q, t \mid q_1, \ldots, q_k)$ has an integrable first moment,

$$C_1(\hat{k}, t \mid q_1, \ldots, q_k) = -ie \int dq\, q\rho_T(q, t \mid q_1, \ldots, q_k) = 0 \tag{3.37}$$

is the time-dependent generalization of the static dipole sum rule (3.33).

We can also recover from (3.19) the small-k behavior of the charge current correlation $d(k, t)$. The charge sum rule (3.24) implies that $d(k=0, t)=0$. Moreover, when we multiply (3.25) by v_1 with $U=(0, v_1)$ and integrate it on v_1, we find that the microscopic motion of $d(k, t)$ at the order $|k|$ is the same as formula (2.27), which was obtained from the macroscopic equation (2.9). An explicit expression of $d_0(\hat{k}, t \mid U)$ for a general U can also be obtained by specifying the appropriate initial conditions.

3.2.2. The Structure Function. According to (3.16), (3.17) and (3.20), $\tilde{S}(k, t)$ and $\tilde{d}(k, t)$ obey the equations

$$\frac{\partial}{\partial t} \tilde{d}(k, t) = \frac{e}{m} \tilde{d}(k, t) \wedge B - i\omega_p^2 \frac{k}{|k|^2} \tilde{S}(k, t) \tag{3.38a}$$

$$+ ie \int dv\, (k \cdot v)\, v\tilde{\rho}_T(k, v, t \mid 0) \tag{3.38b}$$

$$+ \frac{e^3}{m} \int dq \int dq'\, e^{-ik \cdot q} F(q-q')\, \rho_T(q, q', t \mid 0) \tag{3.38c}$$

$$\frac{\partial}{\partial t} \tilde{S}(k, t) = -ik \cdot \tilde{d}(k, t) \tag{3.39}$$

Since $\tilde{d}(k=0, t)=0$, we assume that for small $|k|$,

$$\tilde{d}(k, t)=d_1(\hat{k}, t)|k|+o(|k|) \tag{3.40}$$

One concludes from (3.39) that $\tilde{S}(k, t)$ must be of the order of $|k|^2$. We set

$$\tilde{S}(k, t)=S_2(\hat{k}, t)|k|^2+o(|k|^2) \tag{3.41}$$

and

$$\int dq \int dq' \, |q| \, |\rho_T(q, t \,|\, q', 0)| < \infty \tag{3.42}$$

The assumptions (3.41) and (3.42) are also compatible with the $|q|^{-5}$ decay of the charge–charge correlations found in Section 2.

The term (3.38b) is $o(|k|)$, since (3.18) together with the charge sum rule (3.24) implies

$$\begin{aligned} \tilde{\rho}_T(k=0, v, t \,|\, 0) &= \int dq \, \rho_T(q, v, t \,|\, 0) \\ &= \int dq \, \rho_T(-q, -t \,|\, 0, v)=0 \end{aligned} \tag{3.43}$$

With the change of variable $q' \to q'+q$ and the invariance under translations, the term (3.38c) can be written as

$$\frac{e^3}{m} \int dq' \, F(-q') \int dq \, e^{-ik \cdot q} \rho_T(-q, -t \,|\, q', 0) \tag{3.44}$$

It follows from (3.42), the charge sum rule (3.24), and the dipole sum rule (3.37) that this term is also $o(|k|)$.

In the limit $|k| \to 0$, Eqs. (3.38) and (3.39) reduce to

$$\frac{\partial}{\partial t} d_1(\hat{k}, t)=\frac{e}{m} d_1(\hat{k}, t) \wedge B - i\omega_p^2 \hat{k} S_2(\hat{k}, t) \tag{3.45}$$

$$\frac{\partial}{\partial t} S_2(\hat{k}, t)= -i\hat{k} \cdot d_1(\hat{k}, t) \tag{3.46}$$

which have the same structure as (3.27) and (3.28). The solution is therefore of the form (3.32). Since $\tilde{S}(k, t)$ is even in time, one needs only the two initial conditions

$$S_2(\hat{k}, t=0)=1/4\pi\beta \tag{3.47}$$

(the Stillinger–Lovett second moment condition) and

$$\frac{\partial^2}{\partial t^2} S_2(\hat{k}, t=0) = -i\frac{e}{m}\hat{k}\cdot d_1(\hat{k}, t=0) \wedge B - \omega_p^2 S_2(\hat{k}, t=0)$$

$$= -\frac{\omega_p^2}{4\pi\beta} \tag{3.48}$$

because $d_1(\hat{k}, t=0) = 0$ [see (2.27)].

One recovers then that $\tilde{S}(k, t)$ is given by the formula (2.20) at the order $|k|^2$.

3.3. Inhomogeneous Plasma

When the OCP is not uniform, Fourier transforms are not well adapted, because of lack of translation invariance. Our starting point is again Eqs. (3.9) and (3.10), but the configurational integrals occurring in this subsection are formal. They are in general not absolutely convergent and their meaning has to be specified in each particular case. As in (3.20) and (3.21), we cast the equations of motion in a form analogous to the macroscopic ones by singling out some microscopic correlations. For this purpose, we set

$$\int dq' \frac{1}{|q-q'|} S(q', t \mid q_1) = \phi(q, q_1, t) \tag{3.49}$$

$$e^2 \int dv\, v\rho_T(q, v, t \mid q_1) = d(q, q_1, t) \tag{3.50}$$

One gets from (3.9) and (3.10) with the choice $f(q) = 1/|q|$

$$\frac{\partial}{\partial t} d(q, q_1, t) = -\frac{1}{4\pi}\omega_p^2(q)\nabla_q\phi(q, q_1, t) + \omega_c d(q, q_1, t) \wedge \hat{b}$$

$$+ R_1(q, q_1, t) \tag{3.51}$$

$$\frac{\partial}{\partial t}\phi(0, q_1, t) = \int dq \left(\nabla \frac{1}{|q|}\right) d(q, q_1, t) \tag{3.52}$$

$R_1(q, q_1, t)$ involves the three-body correlation, the electric field due to the inhomogeneity, and a kinetic energy term:

$$R_1(q, q_1, t) = \frac{e^3}{m} E(q)\,\rho_T(q, t \mid q_1) - e^2 \int dv\, v(v\cdot\nabla_q)\,\rho_T(q, v, t \mid q_1)$$

$$+ \frac{e^4}{m}\int dq'\, F(q-q')\,\rho_T(q, q', t \mid q_1) \tag{3.53}$$

It is shown in the Appendix that (3.51) and (3.52) can be combined into the single equation

$$\omega^2(\omega^2-\omega_c^2)\sum_{\alpha,\beta}\int dq\left[\varepsilon_{\alpha\beta}(q,\omega)\nabla_\alpha\frac{1}{|q|}\right]\nabla_\beta\phi(q,q_1,\omega)=R_2(q_1,\omega) \quad (3.54)$$

In (3.54), ω is the time Fourier transform variable. The tensor $\varepsilon_{\alpha\beta}(q,\omega)$ has the same form as in (2.29), but with the local plasmon frequency $\omega_p^2(q)=4\pi(e^2/m)\rho(q)$ defined in terms of the actual particle density $\rho(q)$. The $R_2(q_1,\omega)$ incorporates all the contributions coming from $R_1(q_1,t)$.

The equivalent of the long-wavelength limit amounts to performing the q_1 integral on (3.54) (with an appropriate limiting procedure if it is not absolutely convergent). If it is possible to show that

$$\int dq_1\, R_2(q_1,\omega)=0 \quad (3.55)$$

then (3.54) reduces to an equation involving only $\phi(q,q_1,\omega)$:

$$\omega^2(\omega^2-\omega_c^2)\int dq_1\int dq\sum_{\alpha,\beta}\varepsilon_{\alpha\beta}(q,\omega)\left(\nabla_\alpha\frac{1}{|q|}\right)\nabla_\beta\phi(q,q_1,\omega)=0 \quad (3.56)$$

When the plasma is homogeneous, one can check from the results of Section 3.2 that (3.55) holds (see Appendix). Then, writting (3.56) in Fourier space, it becomes identical to the fourth-order differential equation (3.31) for $S_2(\hat{k},t)$. When $B=0$, one can also derive the sum rule (1.2) from (3.56). For a general inhomogeneous plasma without symmetries and $B\neq 0$, we have not been able to draw definite conclusions from (3.54) because of the lack of information about the asymptotic behavior of the correlations in space.

Let us, however, show how it is possible to recover the special class of inhomogeneous plasmas discussed in Section 2.3. Choosing $f(q)=1$ in (3.10), one notes that the charge sum rule (3.24) is true. According to (2.38), we always interpret the q_1 integral as the "long-cylinder" limit. Now, because of the charge sum rule

$$\nabla_q^2\int dq_1\,\phi(q,q_1,\omega)=-4\pi\int dq'\,S(q',\omega\,|\,q_1)=0 \quad (3.57)$$

Thus, $\int dq_1\,\phi(q,q_1,\omega)$ is harmonic in q, and hence constant with respect to q (assuming boundedness at infinity):

$$\int dq_1\,\phi(q,q_1,\omega)=\int dq_1\,\phi(0,q_1,\omega)=\lim_{R\to\infty}\lim_{L\to\infty} I_A(\omega) \quad (3.58)$$

Arguments are given in the Appendix to show that (3.55) holds. Then, again interpreting the q_1 and q integrals in (3.56) as "long-cylinder limits," and assuming that they can be permuted, one gets, after an integration by parts,

$$\omega^2(\omega^2-\omega_c^2)\int dq \sum_{\alpha\beta} \nabla_\beta \left[\varepsilon_{\alpha\beta}(q,\omega) \nabla_\alpha \frac{1}{|q|} \right] \int dq_1\, \phi(q, q_1, \omega)$$

$$= \omega^2(\omega^2-\omega_c^2)\int dq \sum_{\alpha\beta} \nabla_\beta \left[\varepsilon_{\alpha\beta}(q,\omega) \nabla_\alpha \frac{1}{|q|} \right] \lim_{R\to\infty} \lim_{L\to\infty} I_\Lambda(\omega) = 0 \quad (3.59)$$

Transforming the q integral into an integral on the surface $\partial\Lambda$ of the cylinder Λ, we have

$$\int_\Lambda dq \sum_{\alpha\beta} \nabla_\beta \left[\varepsilon_{\alpha\beta}(r,\varphi) \nabla_\alpha \frac{1}{|q|} \right] = \int_{\partial\Lambda} \sum_{\alpha,\beta} \varepsilon_{\alpha\beta}(r,\varphi) \left(\nabla_\alpha \frac{1}{|q|} \right) d\sigma_\beta \quad (3.60)$$

The bases of the cylinder do not contribute in the limit $L \to \infty$, and, as $R \to \infty$, we are left with

$$\lim_{R\to\infty} R \int_0^{2\pi} d\varphi\, \varepsilon_\perp(R,\varphi) \int_{-\infty}^{+\infty} dz \frac{\partial}{\partial r} \frac{1}{(r^2+z^2)^{1/2}} \bigg|_{r=R} = 4\pi\bar{\varepsilon} \quad (3.61)$$

Here $\bar{\varepsilon}$ is given by (2.39) and (2.40), since by neutrality the particle density $\rho(q) = \rho(R, \varphi)$ tends to the asymptotic background density $\rho_\infty(\varphi)$ as $R \to \infty$.

Introducing (3.61) in (3.59), one obtains finally

$$\omega^2(\omega^2 - \bar{\omega}_p^2 - \omega_c^2) \lim_{R\to\infty} \lim_{L\to\infty} I_\Lambda(\omega) = 0 \quad (3.62)$$

Coming back to the time variable and solving the differential equation with the appropriate initial conditions gives (2.41).

3.4. Dipole Sum Rules

The dipole sum rule of Section 2.4 can be recovered from a simple relation between the semi-infinite and bulk structure functions. A semi-infinite plasma bounded by a plane wall $z = 0$ is subjected to a uniform magnetic field having an angle θ with the normal to the wall. If $S^\infty(q, t) = S^\infty(q, t \mid 0)$ is the corresponding bulk structure function, one establishes along the same lines as in (3.52) and (3.53) in Section 3.3 of Ref. 4 that

$$\int_0^{+\infty} dz_1 \int_{z>0} dq\, z S(q, t \mid z_1) = \int_0^{+\infty} dz_1 \int_{z>0} dq\, (z - z_1)\, S^\infty(q, t \mid z_1) \quad (3.63)$$

This holds under the assumption that the difference $z(S(q, t \mid z_1) - S^{\infty}(q, t \mid z_1))$ is jointly integrable in q and z_1, and that the charge sum rule holds for the semi-infinite system. Since $S^{\infty}(q, t)$ is $O(1/|q|^5)$, its Fourier transform $\tilde{S}(k, t)$ is continuously differentiable, and (3.63) can be written as

$$\int_0^{+\infty} dz_1 \int_{z>0} dq\, zS(q, t \mid z_1) = \frac{i}{2\pi} \int_0^{+\infty} dz \int dl\, e^{il(z-z_1)} f(l) \qquad (3.64)$$

with

$$\begin{aligned} f(l) &= -i \int dq\, zS^{\infty}(q, t)\, e^{-ilz} \\ &= \frac{d}{dk_z} \tilde{S}^{\infty}(k_x, k_y, k_z, t)\Big|_{k_x = k_y = 0, k_z = l} \end{aligned} \qquad (3.65)$$

The rotational invariance of $S^{\infty}(q, t)$ implies that $f(0) = 0$ and $f(-l) = -f(l)$.

Therefore the right-hand side of (3.64) is equal to

$$\begin{aligned} &\frac{i}{2\pi} \int_0^{\infty} dz_1 \left[i\mathscr{P} \int dl \frac{f(l)}{l} e^{-ilz_1} + \pi f(0) \right] \\ &= -\frac{1}{4\pi} \int_{-\infty}^{+\infty} dz_1 \int dl \frac{f(l)}{l} e^{-ilz_1} \\ &= -\frac{1}{2} \lim_{l \to 0} \frac{f(l)}{l} \\ &= -\frac{1}{2} \lim_{l \to 0} \frac{1}{l} \frac{d}{dl} \tilde{S}^{\infty}(0, 0, l, t) \\ &= -\lim_{l \to 0} \frac{1}{l^2} \tilde{S}^{\infty}(0, 0, l, t) \end{aligned} \qquad (3.66)$$

Since B has an angle θ with the z axis, (3.66) is given by (2.20), and the final result is identical to (2.54) [note that $S(q, t \mid q_1) = S(q_1, -t \mid q)$]. The same method enables us to establish the corresponding dipole sum rules for the two-density plasma.

4. CONCLUDING REMARKS

The object of this study is twofold. First, we have obtained exact dynamical sum rules for the one-component plasma, which might be useful in testing approximate theories or comparing with experiments. More generally, we would like to get a better understanding of the relation between macroscopic electrodynamics and statistical mechanics. Here and in Ref. 4, this relation has been made precise in a number of cases. However, none of them include a dissipation mechanism. The analysis of Ref. 4 and of the present paper cannot be extended to the OCP with an (infinite) periodic background where the plasmon mode is damped as a consequence of the coupling of the electrons with the ionic lattice,[8] and it cannot be generalized to multicomponent systems which show dissipation even in the long-wavelength limit because of interparticle collisions.

It should be stressed that the decay in time shown here in (2.23) and (2.48) and resulting from a superposition of oscillations over a continuum of frequencies is not related to any dissipation. For instance, the long-time behavior in (2.23) is governed by the singularity at $\omega = \omega_0 = (\omega_p^2 + \omega_c^2)^{1/2}$, and is easily shown to be proportional to $\cos(\omega_0 t - \pi/4)/t^{1/2}$. However, this damping of the oscillation is only a trivial dispersion effect, since (2.23) was obtained by building a wave packet with undamped plane waves of different frequencies. Similar considerations apply to (2.48).

APPENDIX

To obtain Eq. (3.54), we start from the time Fourier transform of (3.51) and (3.52) (keeping the same symbols d, ϕ,...)

$$-i\omega d(q, q_1, \omega) = -\frac{e^2}{m} \rho(q) \nabla_q \phi(q, q_1, \omega) + \omega_c d(q, q_1, \omega) \wedge \hat{b} + R_1(q, q_1, \omega) \tag{A1}$$

$$-i\omega\phi(0, q_1, \omega) = \int dq \left(\nabla \frac{1}{|q|}\right) \cdot d(q, q_1, \omega) \tag{A2}$$

One multiplies (A2) by $-i\omega$ and substitutes (A1). Repeating this operation three times and using $(d \wedge \hat{b}) \wedge \hat{b} = \hat{b}(d \cdot \hat{b}) - d$, one arrives at

$$\begin{aligned}\omega^4 \phi(0, q, \omega) = \int dq \left(\nabla \frac{1}{|q|}\right) \cdot \Big[& \omega^2 \frac{e^2}{m} \rho(q) \nabla_q \phi(q, q_1, \omega) \\ & + i\omega\omega_c \frac{e^2}{m} \rho(q) \nabla_q \phi(q, q_1, \omega) \wedge \hat{b} \\ & - \omega_c^2 \frac{e^2}{m} \rho(q)\, \hat{b}[\nabla_q \phi(q, q_1, \omega) \cdot \hat{b}] \\ & + i\omega\omega_c^2 d(q, q_1, \omega) \Big] + R_2(q_1, \omega)\end{aligned} \tag{A3}$$

with

$$R_2(q_1, \omega) = -\int dq \left(\nabla \frac{1}{|q|}\right) \cdot \{\omega^2 R_1(q, q_1, \omega) + i\omega\omega_c R_1(q, q_1, \omega) \wedge \hat{b} \\ - \omega_c^2 \hat{b}[R_1(q, q_1, \omega) \cdot \hat{b}]\} \tag{A4}$$

One uses (A2) to eliminate $d(q, q_1, \omega)$ from (A3) and writes $\phi(0, q_1, \omega)$ in the form

$$\phi(0, q_1, \omega) = \int dq\, \delta(q)\, \phi(q, q_1, \omega) \\ = \frac{1}{4\pi} \int dq \left(\nabla \frac{1}{|q|}\right) \cdot \nabla_q \phi(q, q_1, \omega) \tag{A5}$$

With this and rearranging terms, we get (3.54).

Let us show that $\int dq_1\, R_2(q_1) = 0$ for a homogeneous plasma. By inspection of (A4) and (3.53), this will be the case if the following integrals vanish [$E(q) = 0$ when the plasma is homogeneous]:

$$\int dq_1 \int dq \left(\nabla_\alpha \frac{1}{|q|}\right) \int dv\, v_\beta (v \cdot \nabla_q)\, \rho_T(q, v, t \mid q_1) = 0 \tag{A6}$$

$$\int dq_1 \int dq \left(\nabla_\alpha \frac{1}{|q|}\right) \int dq'\, F_\beta(q - q')\, \rho_T(q, q', t \mid q_1) = 0 \tag{A7}$$

Because of translation invariance, (A6) can be Fourier transformed to

$$\int dq_1 \int dq \left(\nabla_\alpha \frac{1}{|q|}\right) \int dv\, v_\beta (v \cdot \nabla_q)\, \rho_T(q - q_1, v_1, t \mid 0) \\ = \lim_{k \to 0} \left(-\frac{4\pi i k_\alpha}{|k|^2}\right) i \int dv\, v_\beta (v \cdot k)\, \tilde{\rho}_T(k, v_1, t \mid 0) \tag{A8}$$

The second factor is identical to the terms (3.38b), which was shown to be $o(|k|)$ [see (3.43)]; therefore (A8) vanishes. In the same way (A7) is equal to $-4\pi i k_\alpha / |k|^2$ times the term (3.38c), which is also $o(|k|)$ [see (3.44)]; thus, (A7) vanishes.

Note that when the plasma is uniform $R_2(q_1) = R_2(r_1, z_1)$ is invariant under the rotations around $\hat{b}$ [$q_1 = (r_1, z_1, \varphi_1)$ with z_1, along $\hat{b}$]. This follows from the form of the integrand of (4) and the fact that $R_1(q, q_1)$ is covariant under such rotations. Therefore $\int dq_1\, R_2(q_1) = 0$ implies

$$\int_0^{+\infty} r_1\, dr_1 \int_{-\infty}^{+\infty} dz_1\, R_2(r_1, z_1) = 0 \tag{A9}$$

To show (3.55) when the plasma is translation invariant only along $\hat{b}$, we argue, as in Section 3.3 of Ref. 4, that we can exchange the q and q_1 integrals occurring in (3.55) and (A4). Let

$$R_2^{(\infty,\varphi_0)}(q_1) = R_2^{(\infty,\varphi_0)}(r_1, z_1)$$

be the quantity (A4) corresponding to a uniform background with density $\rho_\infty(\varphi_0)$. Then (A9) implies

$$\int_0^{2\pi} d\varphi_0 \int_0^{+\infty} r_1\, dr_1 \int_{-\infty}^{+\infty} dz_1\, R_2^{(\infty,\varphi_0)}(r_1, z_1) = 0 \qquad \text{(A10)}$$

We can write [setting $\varphi_0 = \varphi_1$ in (A10)]

$$\int dq_1\, R_2(q_1) = \int_0^{2\pi} d\varphi_1 \int_0^{+\infty} r_1\, dr_1 \int_{-\infty}^{+\infty} dz_1\, [R_2(r_1, z_1, \varphi_1) - R_2^{(\infty,\varphi_1)}(r_1, z_1)] \qquad \text{(A11)}$$

Now $R_2(q_1) - R_2^{(\infty,\varphi_1)}(q_1)$ involves the differences of correlations such as

$$\rho_T(q, q', t \mid q_1) - \rho_T^{(\infty,\varphi_1)}(q, q', t \mid q_1)$$

If the spatial arguments are far apart, both ρ_T and $\rho_T^{(\infty,\varphi_1)}$ decay by clustering; if all arguments tend to infinity in the same direction φ_1, $\rho_T - \rho_T^{(\infty,\varphi_1)}$ vanishes because of the convergence of the correlations to those of the uniform plasma with density $\rho_\infty(\varphi_1)$. We may assume at this point that the decay in the spatial variables enables us to permute the q and q_1 integrals. When the q_1 integral is performed first on (3.53), we have

$$\int dq_1\, R_1(q, q_1, \omega) = 0 \qquad \text{(A12)}$$

because of the charge sum rule (3.24), and thus (3.55) will hold.

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REFERENCES

1. J. H. Malmberg, C. F. Driscoll, and W. D. White, *Physica Scipta* **T2**:288 (1982).
2. J. J. Bollinger and D. J. Wineland, *Phys. Rev. Lett.* **53**:348 (1984).
3. J. L. Lebowitz and Ph. A. Martin, *Phys. Rev. Lett.* **54**:1506 (1985).
4. B. Jancovici, J. L. Lebowitz, and Ph. A. Martin, *J. Stat. Phys.* **41**:941 (1985).
5. L. Landau and E. Lifshitz, *Statistical Physics* (Pergamon, New York, 1980).
6. S. Ichimaru, *Basic Pinciples of Plasma Physics* (Benjamin, Reading, Massachusetts, 1973).
7. Ch. Gruber, J. L. Lebowitz, and Ph. A. Martin, *J. Chem. Phys.* **75**:944 (1981).
8. A. Alastuey and J. P. Hansen, *Europhys. Lett.* **2**:97 (1986).